Audio Signal Processing : IV. Stochastic signal processing

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Framework :

$X[n]$ is a stochastic disrete-time signal

$X[n]$ is real-valued

Stochastic signal : What for ?

Discrete-time stationnary processes : What do we need them for ?

What do we mean by stationarity ?

Strict-sense stationnarity :

$X[n]$: discrete-time stochastic procecss

$$\forall p \in \mathbb{N},\ \forall (n_1, \ldots, n_p) \in \mathbb{Z}^p,\ \forall k \in \mathbb{Z},$$

$$\{X_{n_1}, \ldots, X_{n_p}\} \stackrel{law}{=} \{X_{n_1+k}, \ldots, X_{n_p+k}\}$$

Second-order (wide-sense) stationnarity :

- $X[n]$: discrete-time stochastic process
- $\mathrm{Var}(X[n]) < +\infty$
- $\forall n,\ \mathrm{E}(X[n]) = \mu$
- $\forall n,\ \forall k,\ \ \mathrm{Cov}(X[n], X[n+k]) = R_X[k]$

Theorem

A wide-sense stationnary Gaussian process is strict-sense stationnary

Problem : We want to estimate a deterministic quantity y that is a function of $\{X[n]\}_n$

The **estimator** is a r. v. Y_N that is a function of $\{X[n]\}_{0 \leq N < N}$

The "quality" of the estimator is often quantified by the MSE

$$MSE(N) = E((y - Y_N)^2)$$

Example

- $y = \mathrm{E}(X[n])$
- $Y_N = \frac{1}{N} \sum_{k=0}^{N-1} X[k]$

Problem : y is esimated using the estimator Y_N

Two important quantities

- **Bias** : $Bias_N = E(y - Y_N)$
  $\longrightarrow$ (asymptotically) unbiased estimator ?
- **Variance** : $Var_N = E(Y_N^2) - E(Y_N)^2$
  $\longrightarrow$ consistent estimator ?

Theorem

$$MSE(N) = Bias_N^2 + Var_N$$

An estimation example (Mean estimation):

- $y = E(X[n])$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1} X[k]$ (ergodicity)

Let's study

- Bias
- Consistency ?

Another estimation example (Covariance estimation):

- $y = R_X[k]$
- (using ergodicity) $Y_N = \frac{1}{N-k} \sum_{n=0}^{N-1-k} X[n]X[n+k]$

Let's study

- Bias
- Consistency ?

Another estimation example (A "better" covariance estimator):

- $y = R_X[k]$
- $Y_N = \frac{1}{N}\sum_{n=0}^{N-1-k} X[n]X[n+k]$

- Asymptotically unbiased estimator
- Consistent estimator (if $R_X[k]$ is decreasing quickly enough)

Definition : it is a **positive** bilinear operator

- $A = \sum_{i=1}^{N} a_i X[i]$
- $B = \sum_{i=1}^{N} b_i X[i]$

$$\mathrm{Cov}(A, B) = \sum_{i,j} a_i R_X[i-j] b_j = a.(R \star b) = a.L(B)$$

$\Longrightarrow$ The associated linear form the convolution operator

$$L(B) = R \star b$$

Problem :

How to define the Fourier transform of a stochastic stationnary process ?

Definition of Power spectrum

$$\hat{R}_X(e^{i\omega}) = \sum_n R_X[n] e^{in\omega}$$

- Why is it real ?
- Why is it positive ?
- What is the inverse Fourier transform ?

An interpretation of the Power spectrum

$\hat{R}_X(e^{i\omega})$ represents the average energy contained (in average) by a realization at frequency ω

$$R_X[k] = \frac{1}{2\pi} \int_0^{2\pi} \hat{R}_X(e^{i\omega}) e^{ik\omega} d\omega$$

An important example : The white noise

- $X[n]$ second-order process
- $R_X[k] = \sigma^2 \delta[k]$
- $\hat{R}_X(e^{i\omega}) = 1$

What Estimator for the power spectrum ?

A "natural" estimator could be the **Periodogram**

$$\tilde{\hat{R}}_X(e^{i\omega}) = |\sum_{n=0}^{N-1} X[n]e^{-in\omega}|^2$$

?

A consistent estimator for the power spectrum : The averaged periodogram

$$\tilde{\hat{R}}_X(e^{i\omega}) = \frac{K}{N-1} \sum_{n=0}^{(N-1)/K-1} \left| \sum_{k=nK}^{(n+1)K-1} X[k] e^{-ik\omega} \right|^2$$

Towards the Convolution Theorem

Let $\{X[n]\}_n$ a discrete-time second order stationary process with $\mathrm{E}(X[n]) = 0$ then

- $\forall h \in l^1$, we define $h \star X[n] = \sum_k h[n-k]X[k]$
- $\mathrm{E(h \star X[n])} = \sum_k h[n-k]\mathrm{E}(X[k]) = 0$
- $\forall h \in l^1$, $\forall g \in l^1$, one gets

$$\forall n, n', \quad \mathrm{Cov}(h \star X[n]), g \star X[n']) = R_X \star g \star \tilde{h}[n'-n]$$

- $\forall h \in l^1$, one gets

$$\forall n, k, \quad \mathrm{Cov}(h \star X[n]), h \star X[n+k]) = R_X \star h \star \tilde{h}[k]$$

The Convolution Theorem

Let $\{X[n]\}_n$ a discrete-time second order stationary process with $\mathrm{E}(X[n]) = 0$ then $\forall h \in l^1$, one gets

$$\hat{R}_{h\star X}(e^{i\omega}) = |\hat{h}(e^{i\omega})|^2 \hat{R}_X(e^{i\omega})$$

Property (filtre passe bande)